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CANADA

OPERATIONAL RESEARCH AND ANALYSIS ESTABLISHMENT

DIRECTORATE OF MATHEMATICS AND STATISTICS

DMS STAFF NOTE NO. 3/83

A PRIORITY RANKING MODEL FOR DLR

by

S. ISBRANDT

OTTAWA, CANADA SEPTEMBER, 1983

Introduction

In a preceding Staff Note (Reference 1), several possible methods for constructing a priority ranking of multiple attribute alternatives were presented. One of the methods based on fuzzy sets (Reference 2) was used to form rankings with project information supplied by DLR, on a trial basis. This note provides a description of both the method and the trial.

The Need for Ranking within DLR

DLR essentially maintains a "want list" (Long Term Plan) of projects of various types. These projects are quite diverse: replacements for or additions to existing land forces equipment, upgrades of existing equipment, requirements for acquistion of items which are totally new, et cetera. DLR's justification (through Statements of Requirement, Program Development Proposals, and Program Change Proposals) of the projects on the "want list" may have a major impact on whether they eventually receive funding and are implemented.

DLR has a problem in that there is neither sufficient technical manpower available to allow a most thorough justification for every project, nor is there sufficient funds for every project. Thus, a priority ranking of the

projects would provide guidance in allocating the limited manpower and funds available to those projects which, in some overall sense, are most desirable.

Difficulties Associated with Priority Ranking

One of the main difficulties associated with creating a priority ranking of DLR projects is that they are not distinctly comparable. They may be desirable for very different reasons. One project might provide for replacement of a weapon system which is at the end of its life cycle and is becoming expensive to maintain, another project might provide for replacement equipment which is safer to use than equipment currently used, and another project might contribute to the fulfilling of an international commitment of some sort. The factors or attributes associated with the projects are quite varied. In one fashion or another, either explicitly or implicitly, a trade-off of the various attributes must be made by decision makers in order to produce a priority ranking.

Another major difficulty associated with DLR's ranking of their "want list" is that input from several sections must be combined. The final ranking has to be a consensus of contributing individuals, so the personal preferences of any one section does not unduly affect the ranking.

Basically, DLR's requirement is for a method which constructs a priority ranking of more than 100 multiple attribute alternatives, based on inputs from 5 or more decision makers. It was desirable that DLR have a ranking method which can form a consensus based on information submitted independently.

Overview of Ranking Method

The implementation of a prototype ranking method is now described in this and following sections. Every project is rated by decision makers on a number of attributes, on a scale from 0, which is least favourable, to 10, which is most favourable. These ratings must be consistent with guidelines specified for each attribute. If an individual is not certain of the exact value for a specific attribute rating, he can use an interval to indicate a range of acceptable values. For instance, if he could not decide amongst the rating values of 3, 4, or 5, he could choose the interval from 3 to 5. The individual attribute ratings are subsequently combined into consensus attribute ratings.

Each attribute has a weight, which is its relative importance compared to the rest of the attributes. These weights were determined from pairwise comparisons supplied by representatives from DLR and DLA. From the comparisons, attribute weights were derived for each representative, and these are averaged for use in the ranking procedure.

For each project, the consensus attribute ratings are compared with the corresponding attribute weights. Two overall ratings based on the results of the comparison are computed, one from the results common to all raters, the other from those common to at least half the raters. Then, a single number which is a project's figure of merit is calculated. (See Figure 1.)

The figures of merit for the projects are compared, and an overall priority ranking is made. Those projects which have figures of merit that are close together are grouped into clusters, to indicate that they are essentially equal in rank.

In the DLR trial, a list of 25 alternatives were rated on 10 attributes, and the method was applied to generate a priority ranking. The results presented in this note were generated using prototype computer routines written in APL. Mr. R.Y. Lian of DMS has since written a FORTRAN program which can be used to implement the method.

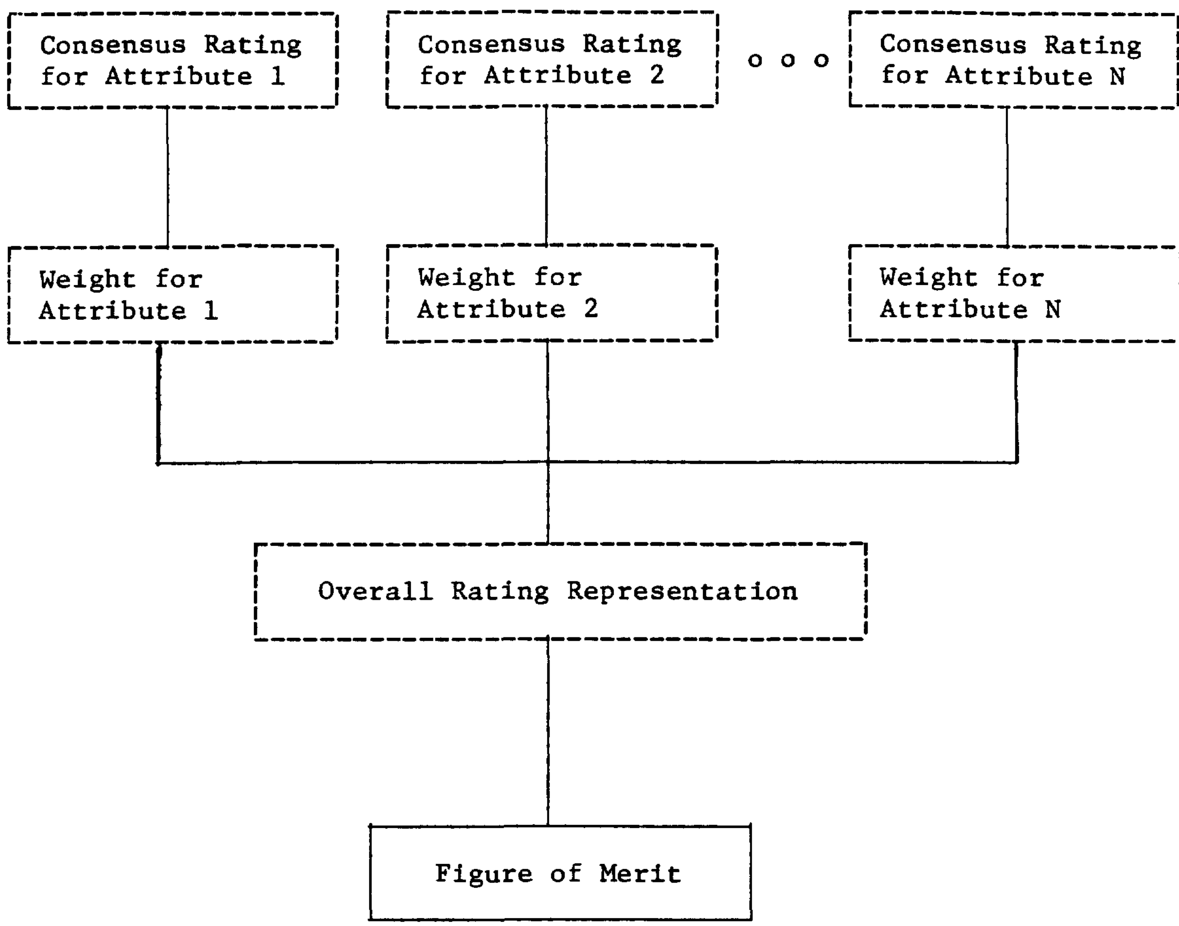

FIGURE 1: Derivation of a Project Figure of Merit

Attribute Rating

Each of the 25 projects were rated according to the 10 attributes shown in Figure 2, by four individuals within DLR. In making these ratings, they used the attribute rating scales shown in Annex A as a guide. Each individual completed a sheet similar to Figure 3 for each project, indicating either a number rating or an interval rating (to accomodate uncertainty) for each attribute. For purposes of further analysis on the computer, these ratings were interpreted as pairs of numbers (a lower bound and an upper bound) as shown in Figure 4.

A computer program to implement the ranking method described in this note was developed by Mr. R.Y. Lian within DMS, and will be documented separately. Most of the input to the computer program consists of contributing individuals' ratings, as described in the previous paragraph. For instance, in the trial for DLR, 4 individuals rating each of 10 attributes on 25 projects resulted in 4x10x25 = 1000 pairs of numbers.

The computer program, after accepting the attribute rankings, produces consensus ratings. In the case of the trial for DLR, four pairs of numbers representing attribute ratings for a particular project are combined to produce a consensus rating for that attribute of that project.

Availability

Cost

Economic Nationalism

External Direction

International Commitment

Life Expectancy

Operational Importance

Safety

Status of Staff Action

Urgency

FIGURE 2 - Attributes

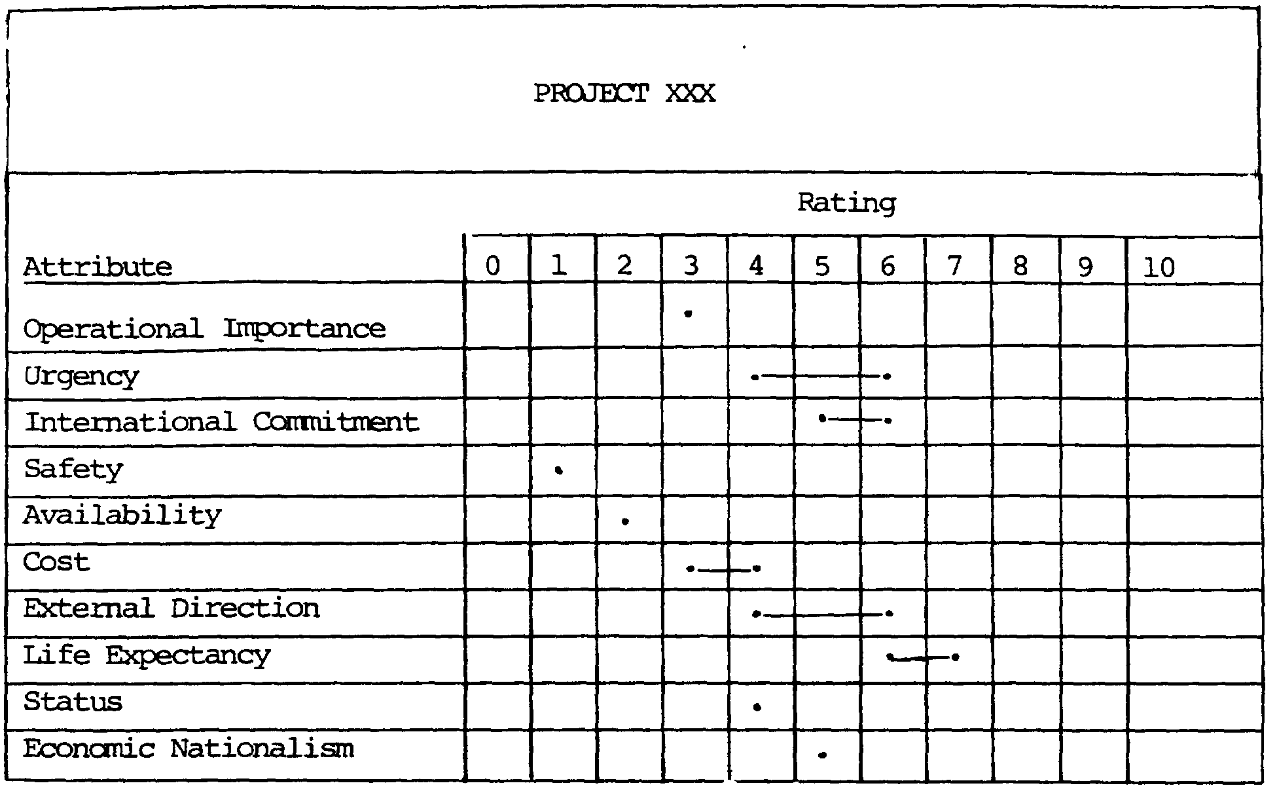

FIGURE 3: Example Rating for Hypothetical Project

Operational Importance	(3,3)
Urgency	(4,6)
International Commitment	(5,6)
Safety	(1,1)
Availability	(2,2)
Cost	(3,4)
External Direction	(4,6)
Life Expectancy	(6,7)
Status	(4,4)
Economic Nationalism	(5,5)

FIGURE 4: Pairs for Example Rating

An example of the formation of a consensus rating is given in Figure 5, with rating intervals represented by lines of X's. A consensus rating is expressed on two levels. A 100% level indicates an interval which is common to all four individuals. (In practice, if there is no such common agreement, an average is used. Also, if there are many decision makers, then any interval common to, say, 80% of them might be used as an approximation to the 100% level.) A 50% level indicates an interval which is common to at least half of the individuals. (Again, in practice if there is no such common area, an average is used). Also, if there are two distinct intervals which are common to at least half of the individuals they are joined to form one long interval. If need be, a 50% interval is extended to include a 100% level which was calculated as an average, since the ranking method assumes the 100% level to be contained within the 50% level.

An example showing the formation of consensus attribute ratings for a project is shown in Figure 6, with intervals specified as pairs of numbers.

	0	1	2	3	4	5	6	7	8	9	10
Individual 1						X	X	X			
Individual 2						X					
Individual 3				X	X	X	X				
Individual 4				X	X	X					
Consensus 100%						X					
50%				X	X	X	X				

<u>FIGURE 5: Formation of Consensus Rating for an Attribute</u>

Attribute	Individual 1 Rating	Individual 2 Rating	Individual 3 Rating	Individual 4 Rating	Consensus 100%	50%
Op Imp	(6,8)	(6,8)	(5,7)	(6,7)	(6,7)	(6,8)
Urgency	(5,5)	(7,8)	(5,6)	(5,5)	(5.75,5.75)	(5,5.75)
Int Com	(6,7)	(6,7)	(4,6)	(6,6)	(6,6)	(6,7)
Safety	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)
Avail	(5,7)	(4,6)	(7,8)	(6,7)	(6.25,6.25)	(5,7)
Cost	(7,7)	(6.5,6.5)	(6,7)	(6,6)	(6.5,6.5)	(6,7)
Ext Dir	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)
Life Exp	(7,9)	(9,9)	(7,8)	(8,9)	(8.13,8.13)	(7,9)
Status	(2,3)	(2,2)	(3,4)	(2,2)	(2.5,2.5)	(2,3)
Ec Nat	(1,2)	(5,5)	(5,7)	(1,1)	(3.63,3.63)	(1,5)

<u>Example</u>: The Urgency 100% interval must be calculated as an average: (5 + 7.5 + 5.5 + 5) ÷ 4 = 5.75. The Urgency 50% interval comprises any overlap common to 2 or more individuals (the interval (5,5), and is extended to include the 100% interval, giving (5, 5.75)).

FIGURE 6: <u>Example of Formation of Consensus Ratings for a Hypothetical Project</u>

Attribute Weights

Although the ranking method developed in DMS by the author allows the specification of attribute weights at 100% and 50% levels, in the DLR trial of the ranking method specific weights were generated for each attribute. The weights were formed using the Analytic Hierarchy Process, described in Reference 3.

A number of representatives from DLR and one from DLA were given sheets similar to FIGURE 7. On the sheets they indicated preference strengths of one attribute versus another, for each pair of attributes. For instance in Figure 7 a check mark on the first line in the third space would indicate that an individual considers "Operational Importance" to strongly dominate "Urgency", when judging the various projects. (Philosophically, the attribute weights are considered to be independent from the specific project to be ranked. Practically, the representatives filling out the preference charts probably cannot help having them in mind.)

These sheets were analyzed according to methods presented in Reference 3 and individual sets of attribute weights were calculated. These weights were returned to the individuals who filled out the sheets for feedback. For the most part, the weights were considered satisfactory. Most

	Absolutely Dominates	Very Strongly Dominates	Strongly Dominates	Weakly Dominates	Equal	Weakly Dominated By	Strongly Dominated By	Very Strongly Dominated By	Absolutely Dominated By	
Op Imp										Urgency
Op Imp										Int Com
Op Imp										Safety
Op Imp										Avail
Op Imp										Cost
Op Imp										Ext Dir
Op Imp										Life Exp
Op Imp										Status
Op Imp										Ec Nat
Urgency										Int Com
Urgency										Safety
o o o										o o o

FIGURE 7: Pairwise Comparison Chart

suggested amendments were minor, with a few large (but not unrealistic) suggestions for amendment. The revised individual weights were averaged to form a consensus set of weights, shown as FIGURE 8.

The weights shown in FIGURE 8 were used as upper and lower interval values, at both the 100% and 50% levels, as input to the DMS computer program.

Attribute	Weight
Availability	10%
Cost	9
Economic Nationalism	9
External Direction	6
International Commitment	12
Life Expectancy	5
Operational Importance	21
Safety	14
Status of Staff Action	5
Urgency	9
	100%

FIGURE 8 - Attribute Weights

As illustrated in FIGURE 1, for each project the consensus attribute ratings are combined with the attribute weights in order to produce an overall rating representation. The manner in which this is done (adapted from Reference 2) is described in Annex B.

The overall rating representation for a project consists of a 100% level interval, and a 50% level interval.

A figure of merit for the project (actually a centroid of the interval representation) is calculated as

$$\frac{Y_1 A_1 + Y_2 A_2}{A_1 + A_2}$$

where Y_1 is the midpoint of 100% interval,

Y_2 is the midpoint of 50% interval,

A_1 is the area of 100% interval (i.e. [.5][U-L] where "U" is upper end of 100% interval and "L" is lower end of 100% interval),

A_2 is the area of 50% interval. (i.e. [.5][U-L] where "U" is upper end of 50% interval, and "L" is lower end of 50% interval.)

In FIGURE 9 overall rating representations and the resulting figures of merit are shown for eight hypothetical projects.

	Interval Level	Overall Rating	Figure of Merit (Centroid)
PROJECT 1	100% 50%	(7.8,9.8) (6,10)	8.3
2	100 50	(4,8.1) (2,9.5)	5.9
3	100 50	(2.5,6.4) (0.9,9.2)	4.9
4	100 50	(4.5,8) (2.1,9.9)	6.1
5	100 50	(1.3,5) (.3,7.5)	3.6
6	100 50	(1.9,5.4) (.6,8.3)	4.2
7	100 50	(2.4,6) (1,8.7)	4.6
8	100 50	(1.2,4.9) (0,8)	3.7

FIGURE 9: Derivation of Project Figure of Merit

Priority Ranking

The computer program written by Mr. R.Y. Lian, after accepting individual sets of ratings and consensus attribute weights, generates figures of merit for the various alternatives. These figures of merit are ordered from highest to lowest to create a priority ranking. However, many of the figures of merit may be close together in value, so that it would be unrealistic to attempt to distinguish between them. So, the program allows the user to specify a level of "clustering", where projects which are closest together in value are essentially considered equal. The program also automatically generates some "clustered" answers at different levels to allow comparison with the user specified one.

FIGURE 10 illustrates figures of merit that were produced from DLR trial data, with the projects identified simply as 1 through 25, along with a sample clustering.

CONCLUSION

It must be emphasized that a structured priority ranking method for multiple attribute alternatives, on complicated scenarios such as the one DLR has, provides a first-cut answer. Consideration of "special cases" and fine-tuning must be left to decision - makers, not to the

FIGURE OF MERIT	PRIORITIZED PROJECT LIST (NO CLUSTERING)
6.09	6
5.96	4
5.81	5
5.66	7
5.19	18
5.14	19
4.95	1
4.88	8
4.85	9
4.75	3
4.71	21
4.70	16
4.68	23
4.65	17
4.50	22
4.44	13
4.18	2
4.11	15
4.06	20
3.65	12
3.56	25
3.56	10
3.56	24
3.45	11
2.86	14

AMENDED FIGURE OF MERIT	PRIORITIZED PROJECT LIST (CLUSTERED TO 9 GROUPS)
6.02	4,6
5.73	5,7
5.17	18,19
4.91	1,8,9
4.71	3,16,17,21,23
4.47	13,22
4.13	2,15,20
3.53	10,11,12,24,25
2.86	14

FIGURE 10: Sample Clustered Ranking

computer. The value of the computer produced ranking is only in providing a reasonable starting configuration, which can be further improved by human decision making.

References

1. S. Isbrandt, "Priority Ranking of Projects Having Multiple Attributes", DMS Staff Note No. 3/81, November 1981.

2. Y.M. Cheng and B. McInnis, "An Algorithm for Multiple Attribute, Multiple Alternative Decision Problems Based on Fuzzy Sets with Application to Medical Diagnosis", IEEE Transactions on Systems, Man, and Cybernetics vol SMC-10, No. 10, pp. 645-650, October 1980.

3. T.L. Saaty, "The Analytic Hierarchy Process", McGraw-Hill, New York, 1980.

ACKNOWLEDGEMENT

The author wishes to thank Mr. R.Y. Lian of DMS for implementing the ranking method presented in this note as a computer program for production purposes.

Annex A
To ORAE Staff Note 3/83
Dated September 1983

Attribute Rating Guidelines

(When using the following scales, a decision maker will, if appropriate, choose a rating value which is between two of the described values. Also, the ranking method treats differences of a certain amount as equal, no matter where they occur on a scale. So, for instance, the difference between 1 and 3 on a given scale has the same value as the difference between 7 and 9 on the same scale.)

Operational Importance:

1 - Insignificant contribution to operational capability
2 - Partial contribution to desirable capability
3 - Desirable
4 - Partial contribution to necessary capability
5 - Necessary capability
7 - Partial contribution to essential capability
9 - Essential capability

Urgency:

1 - Implementation date at least two years beyond minimum DPMS staffing milestones and/or implementation could be deferred two or more years

3 - At optimum time in relation to DPMS milestones and up to two years flexibility in implementation date

5 - At minimum leadtime in relation to DPMS milestone and less than two ears flexibility in implementation date

7 - Accelerated staffing required, or at minimum leadtime and no flexibility in implementation

9 - Immediate essential requirement

Cost:

1 -	\$1,000	million
2 -	\$600 to \$1,000	million
3 -	\$300 to \$600	million
4 -	\$100 to \$300	million
5 -	\$45 to \$100	million
6 -	\$30 to \$45	million
7 -	\$15 to \$30	million
8 -	\$5 to \$15	million
9 -	Less than \$5	million

Availability:

1 - Almost certain to have major problems - cost changes, development time, production, coming into service, delivery

3 - 50% or more chance of significant availability problems

5 - 25% - 50% chance of significant availability problems

7 - low risk

9 - almost no risk

External (to DND) Direction:

1 - No specific external direction

5 - Specific equipment also serves DND low operational requirement

7 - DND committed

9 - Specific equipment also serves DND high operational requirement

Life Expectancy:

1 - Can be used almost indefinitely (eg. helmets)

3 - Can be extended very cost effectively

5 - Extension moderately cost effective

7 - Extension marginally cost effective

9 - Finite and no flexibility

Status of Staff Action:

1 - No formal status

2 - DCDS only approval in DSP period or in LTP

3 - "C" Capital

4 - "C" Capital and PCP milestone achievable within DPMS lead-times; or PCP achievable and DCDS has approved PCP staff action in lieu of PDP

5 - PCP available for final circulation and sign-off within six months (which supports target implementation date)

6 - PCP on final circulation and tentative committee(appropriate approval level) scheduled

7 - PCP approved or expected within two months

8 - "B" Capital and TBS milestone firm

9 - "A" Capital

Economic Nationalism:

1 - Directed to Canadian source and lengthy or high risk industry development involved

3 - Uncertain as to impact

5 - Canadian source possible or definite with significant cost but low risk

7 - Canadian source possible or definite and (in full or part) presents no significant additional cost or risk

9 - No delays, cost, or risk related to Canadian/offshore procurement

Safety:

0 - No specific safety implications

2 - Corrects a condition likely to incur bodily injury

4 - Corrects a condition likely to incur permanent disability or a slight risk to life

6 - Corrects a condition with a moderate risk to life

8 - Corrects a condition with a high risk to life

9 - Corrects a condition with a very high risk to life on a large scale

International Commitment:

0 - No commitment exists

2 - Satisfies partially a low-level or moderate commitment

4 - Satisfies completely a moderate commitment

6 - Partially satisfies a NATO Goal or Pri II or III

7 - Partially satisfied a high-level (MND or Gov't) commitment, or NATO Goal of Pri I

8 - Satisfies a NATO Goal of Pri I

9 - Satisfies a high-level (MND or Gov't) commitment.

Annex B
To ORAE Staff Note 3/83
Dated September 1983

CONSTRUCTION OF OVERALL RATING REPRESENTATIONS IN THE FUZZY SET METHOD

The construction of an overall rating representation is explained here by means of an example. The calculation of the overall 100% level upper and lower endpoints for an alternative with 5 attributes, a to e, is shown. Calculation of the 50% level endpoints would be done in similar fashion.

The 100% level intervals for attribute weights and ratings are reproduced below:

TABLE I

	Interval Level	Attribute a	b	c	d	e
Weights	100%	(.2,.4)	(.2,.4)	(.1,.3)	(0,.1)	(.1,.2)
Ratings	100%	(2,4)	(8,10)	(3,5)	(8,10)	(7,9)

The upper endpoint of the overall rating at the 100% level is an expression of the form:

$$\overline{r} = \frac{\sum_{j=1}^{5} w_j r_j}{\sum_{j=1}^{5} w_j}$$

where r_j is the upper endpoint of the jth attribute rating at the 100% level, and w_j is either the lower or upper endpoint of the jth weight at the 100% level. The w_j values are chosen so as to maximize $\bar{r}$, and are determined with a set of computations which are explained presently.

A series of calculations at most equal to the number of attributes is performed. In the first one, $\bar{r}$ is calculated with w_k being the upper endpoint for k corresponding to a highest rating value r_k, and all other w_j values ($j \neq k$) as lower endpoints. In the second calculation, $\bar{r}$ is worked out with w_k and w_i as upper endpoints for those values of k and i which correspond to the two highest rating values r_k and r_i and all other w_j values ($j \neq k$ or i) are lower endpoints. This pattern continues, with successive w_j values for the third-highest, fourth-highest, etc. r_j values being switched from the lower to the higher endpoint. It can be shown (Appendix 1) that the maximum $\bar{r}$ value calculated amongst such successive trials is indeed the maximum $\bar{r}$ value that can be generated by choosing any combination of w_j and r_j values which are within the specified intevals. The calculation results are shown in Table II.

TABLE II

MAXIMUM $\bar{r}$ CALCULATIONS

TRIAL	w_1	r_1	w_2	r_2	w_3	r_3	w_4	r_4	w_5	r_5	$\bar{r}$
1	.2	4	.4	10	.1	5	0	10	.1	9	7.8
2	.2	4	.4	10	.1	5	.1	10	.1	9	8.0
3	.2	4	.4	10	.1	5	.1	10	.2	9	8.1
4	.2	4	.4	10	.3	5	.1	10	.2	9	7.6
5	.4	4	.4	10	.3	5	.1	10	.2	9	7.1

Maximum $\bar{r}$ value is 8.1

TABLE III

MINIMUM $\bar{r}$ CALCULATIONS

TRIAL	w_1	r_1	w_2	r_2	w_3	r_3	w_4	r_4	w_5	r_5	$\bar{r}$
1	.4	2	.2	8	.1	3	0	8	.1	7	4.3
2	.4	2	.2	8	.3	3	0	8	.1	7	4.0
3	.4	2	.2	8	.3	3	0	8	.2	7	4.3
4	.4	2	.4	8	.3	3	0	8	.2	7	4.8
5	.4	2	.4	8	.3	3	.1	8	.2	7	5.1

Minimum $\bar{r}$ value is 4.0

In calculating the lower endpoint of the overall rating at the 100% level, the same expression for $\bar{r}$ is used, but the r_j values are the lower endpoints of attribute ratings, and w_j values are chosen so as to minimize $\bar{r}$.

Again, a number of calculations at most equal to the number of attributes are performed. In the first one, $\bar{r}$ is calculated with w_k being the upper endpoint for k corresponding to a lowest rating value r_k, and all other w_j values ($j \neq k$) as lower endpoints. In the second calculation, $\bar{r}$ is worked out with w_k and w_i as upper endpoints for those values of k and i which correspond to the two lowest rating values r_k and r_i, and all other w_j values ($j \neq k$ or i) are lower endpoints. This pattern continues, with successive w_j values for third-lowest, fourth-lowest, etc. r_j values being switched from the lower to the higher endpoint. It can be shown that the minimum $\bar{r}$ value calculated amongst such successive trials is indeed the minimum $\bar{r}$ value that can be generated by choosing any combination of w_j and r_j values within the specified intervals. The calculation results are shown in Table III.

So, the overall rating at the 100% level is the interval (4.0, 8.1). Interval calculations at the 50% level are done in the same way as the preceding presentation, using the 50% level (weight and rating) upper and lower endpoints.

APPENDIX 1
TO ANNEX B

Hypothesis

The maximum value for

$$\bar{r} = \frac{\sum_{j=1}^{n} w_j r_j}{\sum_{j=1}^{n} w_j}$$

(where the alternative values are generated by choosing either a "low" or a "high" value for each w_j)

can be formed by using "high" w_j values for a set of r_j values which are <u>all</u> greater than or equal to any r_j value paired with a "low" w_j.

i.e. $$\text{Maximum} = \frac{\sum_{j=1}^{m-1} w_j r_j + \sum_{j=m}^{n} w_j r_j}{\sum_{j=1}^{n} w_j}$$

for some m,

where $r_a \geq r_b$ for $r_a \varepsilon \left\{r_1, r_2, \ldots, r_{m-1}\right\}$

and $r_b \varepsilon \left\{r_m, r_{m\ 1}, \ldots r_n\right\}$

and w_j, $1 \leq j \leq m-1$, are "high" values

and w_j, $m \leq j \leq n$, are "low" values

Proof by Contradiction

$$\text{Assume} \quad \max \bar{r} = \frac{\sum w_j r_j}{\sum w_j}$$

occurs with $r_u > r_d$ such that w_u takes the low value w_u^ℓ, and w_d the high value of w_d^h. Then by this assumption:

(a) If we use both low values for w_u and w_d,

i.e. w_u^ℓ and w_d^ℓ, then $\bar{r}_{\ell\ell} < \bar{r}_{max}$

(b) If we use both high values for w_u and w_d,

i.e. w_u^h and w_d^h, then $\bar{r}_{hh} < \bar{r}_{max}$

Let $w_u^h = w_u^\ell + b_u$

$w_d^h = w_d^\ell + b_d \qquad\qquad b_u, b_d > 0$

and let $\sum w_j$ and $\sum w_j r_j$ imply the use of w_u^ℓ and w_d^ℓ

Then for (a)

$$\frac{\sum w_j r_j + b_d r_d}{\sum w_j + b_d} > \frac{\sum w_j r_j}{\sum w_j}$$

Cross multiplying and cancelling as appropriate, we get

$$r_d \sum w_j > \sum w_j r_j$$

and for (b)

$$\frac{\sum w_j r_j + b_d r_d}{\sum w_j + b_d} > \frac{\sum w_j r_j + b_u r_u + b_d r_d}{\sum w_j + b_u + b_d}$$

Cross multiplying gives:

$$\sum w_j \sum w_j r_j + b_d r_d \sum w_j + b_u \sum w_j r_j + b_u b_d r_d + b_d \sum w_j r_j + b_d^2 r_d$$

$$> \sum w_j \sum w_j r_j + b_u r_u \sum w_j + b_d r_d \sum w_j + b_d \sum w_j r_j + b_u b_d r_u + b_d^2 r_d$$

$$\therefore b_u \sum w_j r_j + b_u b_d r_d > b_u r_u \sum w_j + b_u b_d r_u$$

$$\sum w_j r_j + b_d r_d > r_u \sum w_j + b_d r_u$$

Now from the result for case (a), we can replace $\sum w_j r_j$ in this inequality by $r_d \sum w_j$. Hence

$$r_d \left(\sum w_j + b_d\right) > r_u \left(\sum w_j + b_d\right)$$

$$\therefore r_d > r_u$$

which contradicts the assumption. Hence the hypothesis is proved. The proof for the minimum $\bar{r}$ proceeds similarly.

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